INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids **30**: 1109–1117 (1999)

APPLICATION OF DIFFERENTIAL QUADRATURE METHOD TO SOLVE ENTRY FLOW OF VISCOELASTIC SECOND-ORDER FLUID

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SUMMARY

The entry flow of viscoelastic second-order fluid between two parallel plates is discussed. The governing equations of vorticity and the streamfunction are expanded with respect to a small parameter that characterizes the elasticity of the fluid by means of the standard perturbation method. By using the differential quadrature method with only a few grid points, high-accurate numerical solutions are obtained. The numerical results show a lot of the features of a viscoelastic second-order fluid. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: viscoelasticity; second-order fluid; entry flow; differential quadrature method

1. INTRODUCTION

The differential quadrature method (DQM) proposed by Bellman [1,2] has been successfully employed to obtain numerical solutions in engineering and physical science [3]. Because the information on all grid points is used to fit derivatives at grid points in the DQM, it is enough to use only a few grid points to obtain high-accurate numerical solutions. There are many papers discussing the Newtonian viscous fluid by the DQM [4–8]. But, because of the complexity of the constitutive relations of non-Newtonian fluid, there are only a few papers dealing with this problem by means of DQM. This paper attempts to solve numerically the entry flow of a viscoelastic fluid between two parallel plates by the DQM.

In Reference [9], the collocation and Galerkin finite element methods are used to compute the problem of entry flow of viscous fluid and power-law fluid. Reference [11] discusses the slow flow of a viscoelastic second-order fluid through a contraction by the finite element method. Reference [12] studies an entry flow in a circular tube by the boundary layer theory. In this paper, the second-order model and the governing equations on vorticity and the streamfunction are employed. First, the equations are expanded with respect to a small parameter that characterizes the elasticity of the fluid by means of the standard perturbation method. Then, the resulting zero- and first-order approximative equations are numerically solved by the DQM. By using only a few grid points, good numerical solutions for velocity and stress are obtained. The results obtained in this paper are in agreement with those in [9,11,12].

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CCC 0271–2091/99/161109–09$17.50

Received April 1998
Revised August 1998

The governing equations and the boundary conditions of entry flow are given in Section 2. By means of the standard perturbation method, the zero- and first-order approximative equations are given in Section 3. The implementation of the DQM is explained in detail in Section 4. The numerical results are listed in Section 5.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Let us consider the entry flow model between two parallel plates (Figure 1) in which the velocity components $u = U$ (is constant) and $v = 0$ at the entrance; $u = v = 0$ on the plate. Suppose that the flow is in full development at the exit. On the centerline, $\partial u/\partial y = 0$ and $v = 0$ by symmetry.

Let the viscoelastic fluid be the second order model as follows:

$$\sigma = -p\mathbf{I} + \tau = -p\mathbf{I} + \eta_0 \mathbf{A}_1 + \alpha_1 \mathbf{A}_1^2 + \alpha_2 \mathbf{A}_2, \tag{1}$$

in which σ is the stress tensor, p is the hydrostatic pressure, $\mathbf{I}$ is the unit tensor, τ is the deviatoric tensor of stress; $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second Rivlin–Ericksen tensor respectively, η_0 is the viscosity, α_1 and α_2 are the material parameters characterizing the elasticity of fluid. By introducing dimensionless variables

$$x := \frac{x}{H}, \quad y := \frac{y}{H}, \quad u := \frac{u}{U}, \quad v := \frac{v}{U}, \quad p := \frac{H}{\eta_0 U} p, \quad \sigma := \frac{H}{\eta_0 U} \sigma, \quad \tau := \frac{H}{\eta_0 U} \tau,$$

the components of the deviatoric tensor of stress can be written as follows:

$$\tau_{xx} = 2\frac{\partial u}{\partial x} + \beta_1 \left[4\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right] + \beta_2 \left[2u\frac{\partial^2 u}{\partial x^2} + 2v\frac{\partial^2 u}{\partial x\,\partial y} + 4\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \right], \tag{2.1}$$

$$\tau_{yy} = 2\frac{\partial v}{\partial y} + \beta_1 \left[4\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right] + \beta_2 \left[2u\frac{\partial^2 v}{\partial x\,\partial y} + 2v\frac{\partial^2 v}{\partial y^2} + 4\left(\frac{\partial v}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \right], \tag{2.2}$$

$$\tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \beta_2 \left[u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \right], \tag{2.3}$$

where $\beta_1 = \alpha_1 U/\eta_0 H$, $\beta_2 = \alpha_2 U/\eta_0 H$ are dimensionless parameters.

The governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

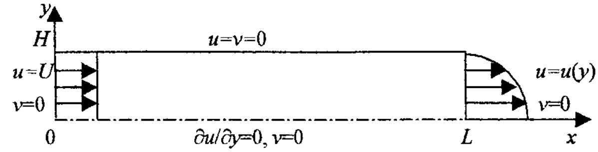

Figure 1. Entry flow model.

$$Re\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}, \tag{3.2}$$

$$Re\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}, \tag{3.3}$$

where $Re=\rho UH/\eta_0$ is the Reynolds number.

Let the streamfunction Ψ and vorticity ω satisfy

$$u=\frac{\partial \Psi}{\partial y}, \qquad v=-\frac{\partial \Psi}{\partial x}, \qquad \omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{4}$$

Substituting (2) into (3) and eliminating pressure p we get

$$Re\left(u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}\right)=\nabla^2\omega+\beta_2\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\nabla^2\omega, \tag{5.1}$$

$$\nabla^2\Psi=-\omega. \tag{5.2}$$

Equations (5.1) and (5.2) are the governing equations of the fundamental unknown variables Ψ and ω. The boundary conditions can be written as

$$\Psi=y, \quad \omega-\frac{\partial v}{\partial x}=0, \quad u=1, \quad v=0 \quad \text{for } x=0,\ 0\leq y\leq 1, \tag{6.1}$$

$$\frac{\partial \Psi}{\partial x}=0, \quad \frac{\partial \omega}{\partial x}=0, \quad \frac{\partial u}{\partial x}=0, \quad v=0 \quad \text{for } x=a,\ 0\leq y\leq 1, \tag{6.2}$$

$$\Psi=0, \quad \omega=0, \quad \frac{\partial u}{\partial y}=0, \quad v=0 \quad \text{for } y=0,\ 0<x<a, \tag{6.3}$$

$$\Psi=1, \quad \omega+\frac{\partial u}{\partial y}=0, \quad u=0, \quad v=0 \quad \text{for } y=1,\ 0<x<a, \tag{6.4}$$

where $a=L/H$.

3. PERTURBATION EXPANSION

If the elasticity of the fluid is slight, β_1 and β_2 can be considered as small parameters. According to [10], we have $\beta_2<0$, $\beta_1>0$ and $\beta_1=-c\beta_2$, $c\approx 1.6$. So we can introduce only one small parameter $\varepsilon=-\beta_2$ in the problem, then $\beta_1=-c\varepsilon$. By expanding ω, Ψ, u and v as

$$\omega=\omega_0+\varepsilon\omega_1, \quad \Psi=\Psi_0+\varepsilon\Psi_1, \quad u=u_0+\varepsilon u_1, \quad v=v_0+\varepsilon v_1, \tag{7}$$

and substituting (7) into (4) and (5), then neglecting the terms of order ε^2, we get zero-order approximative equations

$$Re\left(u_0\frac{\partial \omega_0}{\partial x}+v_0\frac{\partial \omega_0}{\partial y}\right)=\nabla^2\omega_0, \tag{8.1}$$

$$\nabla^2\Psi_0=-\omega_0, \tag{8.2}$$

$$u_0=\frac{\partial \Psi_0}{\partial y}, \qquad v_0=-\frac{\partial \Psi_0}{\partial x}, \tag{8.3}$$

and the first-order approximative equations

$$Re\left(u_0\frac{\partial\omega_1}{\partial x}+v_0\frac{\partial\omega_1}{\partial y}+u_1\frac{\partial\omega_0}{\partial x}+v_1\frac{\partial\omega_0}{\partial y}\right)=\nabla^2\omega_1-\left(u_0\frac{\partial}{\partial x}+v_0\frac{\partial}{\partial y}\right)\nabla^2\omega_0, \tag{9.1}$$

$$\nabla^2\Psi_1=-\omega_1, \tag{9.2}$$

$$u_1=\frac{\partial\Psi_1}{\partial y},\qquad v_1=-\frac{\partial\Psi_1}{\partial x}. \tag{9.3}$$

Obviously, the zero-order approximative equations are the governing equations of an incompressible Newtonian fluid, and the boundary conditions of ω_0, Ψ_0, u_0 and v_0 are same as in (6). The boundary conditions of ω_1, Ψ_1, u_1 and v_1 are the homogeneous form of (6), which means all of the right-hand-sides in Equations (6) should be zeros.

By expanding the deviatoric stress τ as $\tau=\tau^0+\varepsilon\tau^1$, we get the components of the zero-order approximation τ^0 as follows:

$$\tau^0_{xx}=2\frac{\partial u_0}{\partial x},\qquad \tau^0_{yy}=2\frac{\partial v_0}{\partial y},\qquad \tau^0_{xy}=\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}. \tag{10}$$

Obviously, τ^0 is the deviatoric stress of the incompressible Newtonian flow and can be considered as the contribution of viscosity of the fluid. The components of the first-order approximation τ^1 are

$$\tau^1_{xx}=2\frac{\partial u_1}{\partial x}+c\left[4\left(\frac{\partial u_0}{\partial x}\right)^2+\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)^2\right]-\left[2u_0\frac{\partial^2u_0}{\partial x^2}+2v_0\frac{\partial^2u_0}{\partial x\,\partial y}+4\left(\frac{\partial u_0}{\partial x}\right)^2+2\frac{\partial v_0}{\partial x}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)\right], \tag{11.1}$$

$$\tau^1_{yy}=2\frac{\partial v_1}{\partial y}+c\left[4\left(\frac{\partial v_0}{\partial y}\right)^2+\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)^2\right]-\left[2u_0\frac{\partial^2v_0}{\partial x\,\partial y}+2v_0\frac{\partial^2v_0}{\partial y^2}+4\left(\frac{\partial v_0}{\partial y}\right)^2+2\frac{\partial u_0}{\partial y}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)\right], \tag{11.2}$$

$$\tau^1_{xy}=\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}-\left[u_0\frac{\partial}{\partial x}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+v_0\frac{\partial}{\partial y}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+2\frac{\partial u_0}{\partial x}\frac{\partial u_0}{\partial y}+2\frac{\partial v_0}{\partial x}\frac{\partial v_0}{\partial y}\right], \tag{11.3}$$

which can be considered as the contribution of elasticity of the fluid.

4. NUMERICAL METHOD

Placing $N\times M$ uniform grid points (x_i, y_j), $1\le i\le N$, $1\le j\le M$ in the considered domain $0\le x\le a$, $0\le y\le 1$, the derivatives of a function $f(x, y)$ at grid point (x_i, y_j) can be approximated by DQ discretization

$$f_x(i,j)=\sum_{k=1}^{N}A^{(x)}_{ik}f(k,j),\qquad f_{xx}(i,j)=\sum_{k=1}^{N}B^{(x)}_{ik}f(k,j), \tag{12.1}$$

$$f_y(i,j)=\sum_{l=1}^{M}A^{(y)}_{jl}f(i,l),\qquad f_{yy}(i,j)=\sum_{l=1}^{M}B^{(y)}_{jl}f(i,l), \tag{12.2}$$

where $f(i,j)$ denotes $f(x_i, y_j)$, $1\le i\le N$, $1\le j\le M$. The quadrature coefficients $A^{(x)}_{ik}$, $B^{(x)}_{ik}$, $A^{(y)}_{jl}$, $B^{(y)}_{jl}$ can be computed by the formulae given in [4].

Applying (12) to the zero-order approximative equations (8), we get

$$\sum_{k=1}^{N}[Reu_0(i,j)A_{ik}^{(x)}-B_{ik}^{(x)}]\omega_0(k,j)+\sum_{l=1}^{M}[Rev_0(i,j)A_{jl}^{(y)}-B_{jl}^{(y)}]\omega_0(i,l)=0, \tag{13.1}$$

$$\sum_{k=1}^{N}B_{ik}^{(x)}\Psi_0(k,j)+\sum_{l=1}^{M}B_{jl}^{(y)}\Psi_0(i,l)=-\omega_0(i,j), \tag{13.2}$$

$$u_0(i,j)=\sum_{l=1}^{M}A_{jl}^{(y)}\Psi_0(i,l),\qquad v_0(i,j)=-\sum_{k=1}^{N}A_{ik}^{(x)}\Psi_0(k,j), \tag{13.3}$$

where $2\leq i\leq N-1$, $2\leq j\leq M-1$.

The terms $\partial\omega/\partial x$ and $\partial v/\partial x$ denote the boundary conditions of the exit and the entrance respectively. Since these boundary conditions are independent of the values of ω and v at another end, the DQ method is not used and the second-order difference formula is employed for discretization of these derivatives in the boundary conditions of the exit and the entrance.

Because (13) and the discrete boundary conditions are non-linear, an iterative procedure must be adopted to solve (13). If the $(s-1)$th iterative solutions $\omega_0^{(s-1)}$, $\Psi_0^{(s-1)}$, $u_0^{(s-1)}$,$v_0^{(s-1)}$ are known from (13), with $u_0=u_0^{(s-1)}$, $v_0=v_0^{(s-1)}$, and the boundary conditions of ω_0, we can solve ω_0 for the internal points. Then, let $\omega_0^{(s)}=(1-\theta)\omega_0^{(s-1)}+\theta\omega_0$, where ω_0 is the solution of (13.1) and $0<\theta<1$ is a damping factor. From (13.2), with $\omega_0=\omega_0^{(s)}$ and the boundary conditions of Ψ_0, we can get $\Psi_0^{(s)}$. Finally, from (13.3) we can find $u_0^{(s)}$ and $v_0^{(s)}$. Therefore, the unknown boundary values of $\omega_0^{(s)}$ can be obtained from the boundary conditions. This procedure should be carried out successively until $\max|\omega_0^{(s)}-\omega_0^{(s-1)}|\leq e$, where e is an iterative accuracy given previously.

Applying (12) to the first-order approximative equations (9) gives

$$\sum_{k=1}^{N}[Reu_0(i,j)A_{ik}^{(x)}-B_{ik}^{(x)}]\omega_1(k,j)+\sum_{l=1}^{M}[Rev_0(i,j)A_{jl}^{(y)}-B_{jl}^{(y)}]\omega_1(i,l)$$
$$=-Re\left[u_1(i,j)\frac{\partial\omega_0}{\partial x}(i,j)+v_1(i,j)\frac{\partial\omega_0}{\partial y}(i,j)\right]-\left[u_0\frac{\partial}{\partial x}+v_0\frac{\partial}{\partial y}\right]\nabla^2\omega_0(i,j), \tag{14.1}$$

$$\sum_{k=1}^{N}B_{ik}^{(x)}\Psi_1(k,j)+\sum_{l=1}^{M}B_{jl}^{(y)}\Psi_1(i,l)=-\omega_1(i,j), \tag{14.2}$$

$$u_1(i,j)=\sum_{l=1}^{M}A_{jl}^{(y)}\Psi_1(i,l),\qquad v_1(i,j)=-\sum_{k=1}^{N}A_{ik}^{(x)}\Psi_1(k,j), \tag{14.3}$$

where $2\leq i\leq N-1$, $2\leq j\leq M-1$. An iterative procedure is also adopted to solve the discrete system (14).

5. NUMERICAL RESULTS AND DISCUSSION

Using the numerical method mentioned above, the numerical computations are carried out for the entry flow of the viscoelastic second-order fluid with $Re=1$, 5, 10. In the actual computation, we choose $a=4$, $N=21$, $M=11$, $\Delta x=0.2$, $\Delta y=0.1$ and the iterative accuracy $e=10^{-5}$. The convergent numerical solutions are obtained when the damping factor $\theta<0.2$.

Figure 2 shows the typical profiles of component u_0 of the zero-order velocity at different positions between two parallel plates. As a result of viscosity, when x increase, u_0 decreases near the plates and increases near the centerline, the flow tends to full development gradually. As the Reynolds number increases, the required horizontal distance to reach fully developed flow (i.e. entrance length) increases gradually. These are in agreement with [9] in the qualitative

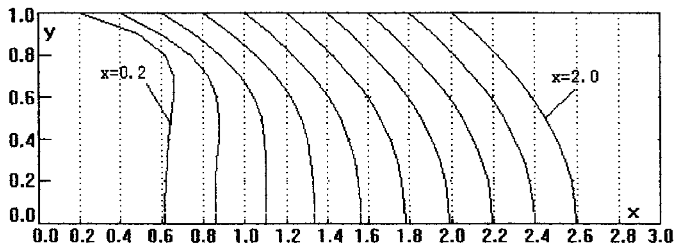

Figure 2. Typical profiles of zero-order velocity component u_0 ($Re = 5$).

Table I. Zero-order velocity components u_0 at the centerline

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6
$Re = 1$	1.07	1.20	1.32	1.41	1.45	1.48	1.49	**1.50**	1.50	1.50	1.50	1.50	1.50
$Re = 5$	1.03	1.14	1.25	1.34	1.41	1.45	1.47	1.48	1.49	**1.50**	1.50	1.50	1.50
$Re = 10$	1.02	1.09	1.17	1.26	1.33	1.38	1.42	1.45	1.46	1.47	1.48	1.49	**1.50**

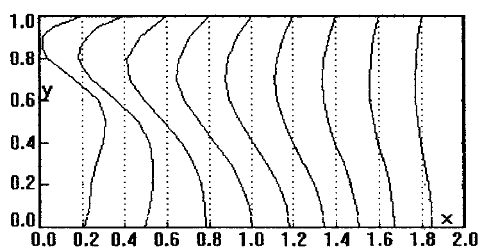

Figure 3. Typical profiles of first-order velocity component u_1 ($Re = 5$).

Table II. Maximum and minimum values of u_1 at different positions ($Re = 5$)

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
Max	0.169	0.199	0.273	0.303	0.271	0.216	0.159	0.111	0.073
Min	−0.278	−0.328	−0.281	−0.230	−0.183	−0.137	−0.097	−0.067	−0.044

analysis. Table I lists the values of u_0 at the centerline with different Reynolds number and shows that the entrance length is about 1.6, 2.0, 2.6 for $Re = 1$, 5, 10 respectively. Although the number of grid points is less and the step is large, accurate numerical solutions are still obtained. It shows that high-accurate numerical solutions can be obtained by the numerical method proposed in this paper.

Figure 3 shows the typical profiles of components u_1 of the first-order velocity, Table II lists the maximum and minimum values of u_1 at the different positions. One can see that the elastic effect makes the horizontal velocity component decrease near the plates and increase near the centerline. This trend takes the velocity to full development more rapidly, i.e. the entrance length of the entry flow is shortened, in agreement with [11,12].

The typical streamlines for the zero- and first-order system are sketched in Figures 4 and 5. The elastic effect of the second-order fluid decreases with development of the flow.

Figure 6 shows the sketch of the components of the deviatoric stress τ^0_{xx}, $\varepsilon\tau^1_{xx}$, τ_{xx} and τ^0_{yy}, $\varepsilon\tau^1_{yy}$, τ_{yy} at the centerline ($\varepsilon = -\beta_2 = 0.1$). The shear rate is small near the centerline, the elasticity of the fluid slightly influences the stress, so the stress of the second-order fluid is close to that of the Newtonian fluid. The sketch of the components of the deviatoric stress τ^0_{xx}, $\varepsilon\tau^1_{xx}$, τ_{xx} and τ^0_{yy}, $\varepsilon\tau^1_{yy}$, τ_{yy} near the wall ($y = 0.9$) are shown in Figure 7 ($\varepsilon = -\beta_2 = 0.1$). The shear rate is large near the wall, the elasticity of the fluid strongly influences the stress, so the stress of the second-order fluid is evidently different to that of the Newtonian case. The first normal stress difference $\tau_{xx} - \tau_{yy}$ can be observed near the wall.

It should be pointed out that, as the Reynolds number increases, the influence of the fluid elasticity upon the velocity field extends, and the influence upon the stress components near

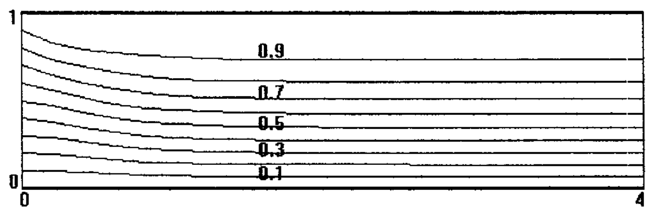

Figure 4. Typical streamlines of zero-order system ($Re = 5$).

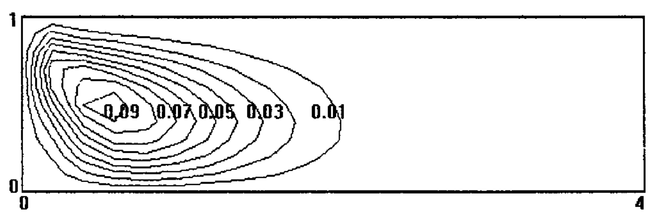

Figure 5. Typical streamlines of first-order system ($Re = 5$).

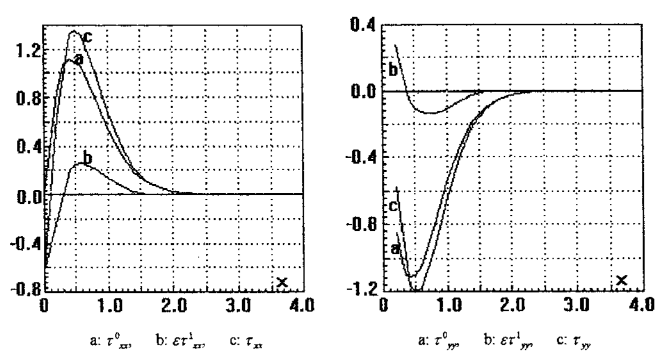

Figure 6. Profiles of stress components at the centerline ($Re = 5$).

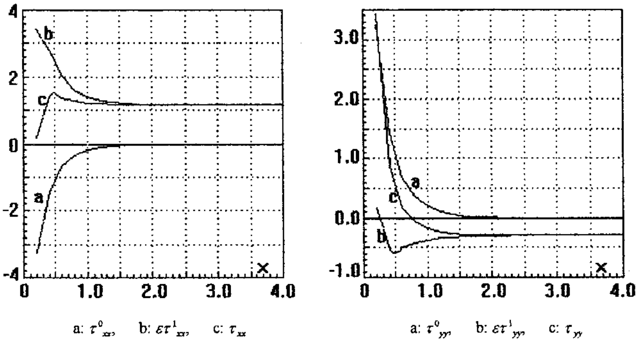

Figure 7. Profiles of stress components near the wall ($Re = 5$).

the entrance increases. But in the fully developed flow domain, the influence upon the stress components does not changes as the Reynolds number increases.

6. CONCLUSION

For the entry flow of viscoelastic second-order fluid between two parallel plates, the governing equations of vorticity and the streamfunction are expanded in terms of a small parameter that characterizes the elasticity of the fluid by using a standard perturbation method. The zero- and first-order approximative equations obtained by the perturbation method are numerically solved by means of the DQM. The high-accurate numerical solutions are obtained with only a few grid points. The numerical results show that the elasticity in the second-order fluid makes the entrance length short, and it influences the stress near the centerline slightly and the stress near the wall strongly.

ACKNOWLEDGMENTS

This paper is supported by the National Natural Science Foundation of China and Shanghai Municipal Doctoral Program Foundation of Institutions of Higher Education.

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